AD A056073

LEVEL (12)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS
BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
Technical Report No. 30		

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
TWO-PIECE VON NEUMANN-MORGENSTERN UTILITY FUNCTIONS	Interim
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Peter C. Fishburn and Gary A. Kochenberger	N00014-75-C-0857

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
The Pennsylvania State University University Park, Pennsylvania 16802	121407

11. CONTROLLING OFFICE NAME AND ADDRESS (Code 613B: SJW)	12. REPORT DATE
Office of Naval Research, Dept. of the Navy 800 N. Quincy Street Arlington, Virginia 22217	June 1978
	13. NUMBER OF PAGES
	30

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution of this document is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

DDC
JUL 7 1978
RECEIVED
F

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Utility functions
Risk aversion
Risk seeking

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Thirty empirically-assessed utility functions on changes in wealth or return on investment were examined for general features and susceptability to fits by linear, power and exponential functions. Separate fits were made to below-target data and above-target data. The usual target was the n-change point.

The majority of below-target functions were risk-seeking, the majority of above-target functions were risk-averse, and the most common composite shape was convex-concave, or risk-seeking in losses and risk-averse in gains. → Next page

DD FORM 1473 1 JAN 73   EDITION OF 1 NOV 65 IS OBSOLETE

20. The least-common composite was concave-concave. Below-target utility was generally steeper than above-target utility with a median below-to-above slope ratio of about 4.8. The power and exponential fits were substantially better than the linear fits. Power functions gave the best fits in the majority of convex below-target and concave above-target cases, and exponential functions gave the best fits in the majority of concave below-target and convex above-target cases.

ACCESSION for

NTIS   White Section
DDC   Buff Section
UNANNOUNCED
JUSTIFICATION

BY
DISTRIBUTION/AVAILABILITY CODES
DIST.   SPECIAL

UNCLASSIFIED

TECHNICAL REPORT NO. 30

TWO-PIECE VON NEUMANN-MORGENSTERN UTILITY FUNCTIONS

TR-30

Peter C. Fishburn
and
Gary A. Kochenberger
College of Business Administration
The Pennsylvania State University
University Park, Pennsylvania 16802

Jun 78   33p.

78 07 03 016

407 732   JOB

INTRODUCTION

Although the von Neumann-Morgenstern expected utility theory [5] [14] [25] pertains to any risky situation in which consequences' probabilities have been assessed, it has received special attention in monetary settings. As long ago as 1728, Gabriel Cramer suggested [4, p. 34] that a gain in present wealth might have utility proportional to the square root of the gain. A few years later, and without prior knowledge of Cramer's contribution, Daniel Bernoulli [3][4] argued that--in the absence of unusual circumstances--the utility of wealth w will be equal to b log (w/a), where a and b are positive constants. After the von Neumann-Morgenstern revival and axiomatization of expected utility, Friedman and Savage [7] proposed a utility of income function for low-income consumer units that has a convex (risk-preferring or risk-seeking) segment surrounded by two concave (risk-averse) segments to explain, among other things, gambling and insurance-buying behavior. Markowitz [20] then modified this by suggesting a four-segment bounded utility function of wealth that is initially convex, then concave, then turns convex again in the vicinity of present wealth, and ends up concave. Additional comments on the Friedman-Savage and Markowitz proposals are made in [1] [11] [18] [22] [26].

In contrast to these armchair proposals, a number of research workers--including Grayson and Swalm [9] [24]--have attempted to assess individuals' utility functions over significant changes in individual or corporate wealth, or in return on investment [10]. This work adopts the proposition [16] that changes in wealth rather than final asset positions govern utility or value judgments and its concomitant choice behavior. The data it has generated are reviewed by Fishburn, Libby and Fishburn, and Kahneman and Tversky [6] [19] [16], and suggest four interesting things. First, there is usually a point t

on the abscissa at which something unusual happens to the individual's utility

function. This target point [6] [19] or reference point [16] is often the

zero-gain point, as in Markowitz's case [20] or in Swalm's data [24], but may

be in the loss region (some wildcatters [9]) or the gain region [10]. Second,

below-target utility is frequently convex [2] [8] [12] [16] [23] [24].

Third, a majority of above-target utility functions are concave, or risk-averse.

And fourth, below-target utility, whether convex, concave or linear, is

almost always steeper than above-target utility. Figure 1 gives a composite

Figure 1 about here

picture of these things. An alternative below-target concave segment is shown

by the dashed curve (cf. [13], Exhibit IX).

The present paper has two main purposes. The first is to reëxamine the

published assessment data to obtain a more precise picture of the generalizations

noted above. The second is to get an idea of the extent to which simple

functional forms yield good fits to the assessment data. In each case we

shall calculate or estimate the best fit for each functional type to the

above-target data and to the below-target data. The parameters of these fits

will then be used to evaluate slope and curvature questions. We shall also

note which functional type provides the best below-target fit and which gives

the best above-target fit. The best-fit criterion used throughout is minimum

least squares.

The next section discusses the functional forms that were fit to the

data. This is followed by remarks on our fitting methodology. The penultimate

section lists the data sources, identifies best fits for each case, and

FIGURE 1
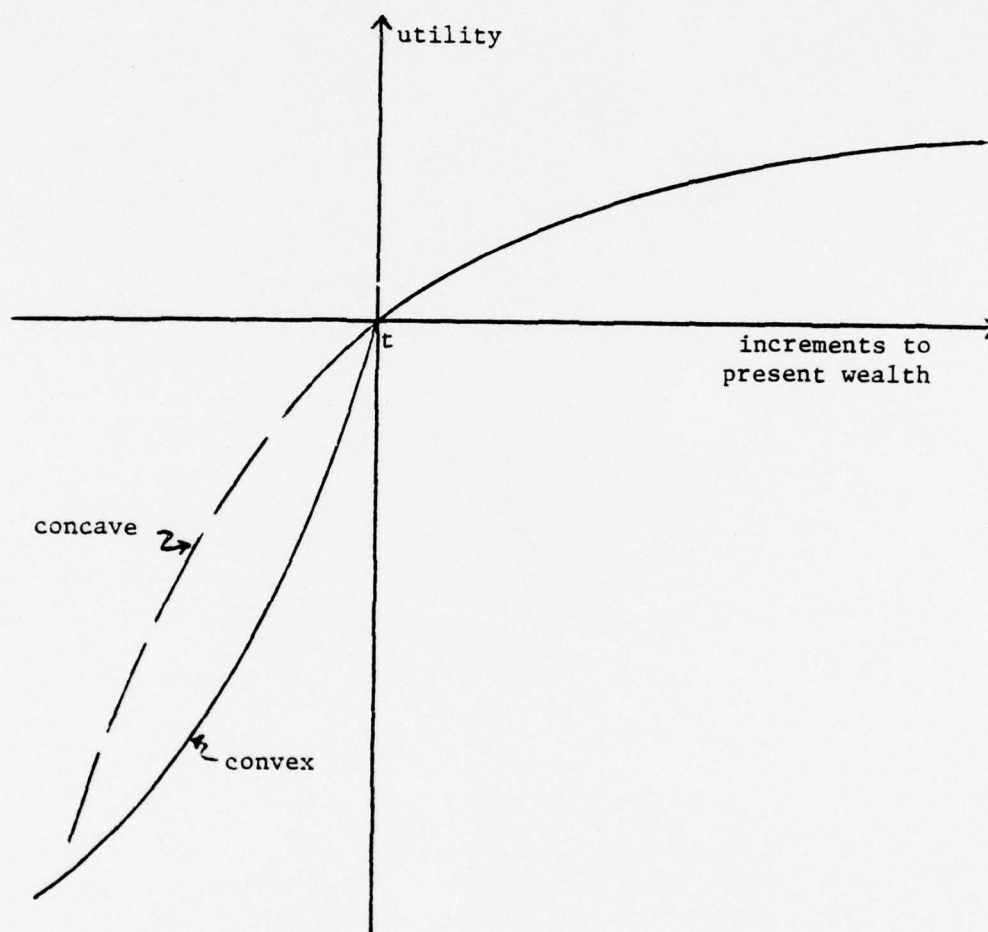
Hypothetical Utility Functions for $ Increments

summarizes salient aspects of these fits. The paper concludes with an overview of main results.

Since the study uses the data available in the published literature on the assessment of von Neumann-Morgenstern utility functions, our sample cannot be regarded as random or unbiased. Nevertheless, utility assessments from many individuals engaged in different pursuits are represented in the data. Consequently, we feel that these data give a reasonable picture of what might be expected in other situations.

FUNCTIONAL FORMS

The data for each case examined later will be transformed linearly so that the transformed target point is $x = 0$ and the utility at the origin is zero. Three simple functions will be fit to the data, and this will be done separately for $x \geq 0$ and $x \leq 0$. The three functions are the linear (L), power (P) and exponential (E) functions. These are defined as follows for $x \geq 0$:

	$u(x), x \geq 0$	$u'(x)$	$u''(x)$
L^+	$cx;\ c > 0$	c	0
P^+	$a_1 x^{a_2};\ a_1 > 0,\ a_2 > 0$	$a_1 a_2 x^{a_2 - 1}$	$a_1 a_2 (a_2 - 1)x^{a_2 - 2}$
E^+	$b_1(1 - e^{-b_2 x});\ b_1 b_2 > 0$	$b_1 b_2 e^{-b_2 x}$	$-b_1 b_2^2 e^{-b_2 x}$

The corresponding functions for $x \leq 0$ are obtained from these by replacing x by $-x$ and multiplying by -1:

	$u(x), x \leq 0$	$u'(x)$	$u''(x)$
L^-	$cx; c > 0$	c	0
P^-	$-a_1(-x)^{a_2}; a_1 > 0, a_2 > 0$	$a_1 a_2 (-x)^{a_2 - 1}$	$-a_1 a_2 (a_2 - 1)(-x)^{a_2 - 2}$
E^-	$-b_1(1 - e^{b_2 x}); b_1 b_2 > 0$	$b_1 b_2 e^{b_2 x}$	$b_1 b_2^2 e^{b_2 x}$

These functions were chosen for their analytical simplicity and economy of parameters, and for their ability to approximate or generate a large variety of specific forms. Moreover, such functions have been used extensively to describe phenomena in the physical and social sciences. We invite readers who feel that other forms may give good fits to the subjective empirical data to compare these with L, P and E.

Although L is the special case of P with $a_2 = 1$, we decided to isolate it for three reasons. First, some of the data plots either above or below $x = 0$ looked nearly linear, in which case L may provide a very good fit; second, the minimum mean squared error (MMSE) for a minimum least-squares L fit offers a useful benchmark against which to compare the MMSE values of the P and E fits; and third, the best L^+ and L^- fits give an indication of the change in slope of the utility function around $x = 0$ that is not unduly confounded with curvature. When $a_2 \neq 1$, it may be noted that the slope of P^+ at 0^+ is either zero $(a_2 > 1)$ or infinity $(a_2 < 1)$, and the slope of P^- at 0^- is either zero $(a_2 > 1)$ or infinity $(a_2 < 1)$.

It should be noted also that while a two-piece linear utility function—which we denote as $L^- L^+$—is risk neutral in the negative region and in the positive region, it is risk neutral for all gambles if and only if it does not change slope at the origin. If the slope of L^- is greater than the slope of L^+,

then gambles that have both negative and positive outcomes exhibit risk

aversion, whereas risk-seeking behavior applies for such gambles when the

slope of L^- is less than the slope of L^+.

For power functions, P^+ is risk-averse ($u''(x) < 0$, or u is concave) if

$a_2 < 1$ and risk-seeking ($u''(x) > 0$, or u is convex) if $a_2 > 1$, and P^- is

risk-averse or concave if $a_2 > 1$ and risk-seeking or convex if $a_2 < 1$. The

picture for gambles that have both negative and positive outcomes is more

complex when u is a two-piece power function, of type $P^- P^+$. For example,

such gambles may exhibit risk-seeking behavior (certainty equivalent of the

gamble exceeds its actuarially fair value) when both P^+ and P^- are risk-averse.

For exponential functions, which require both parameters to have the same

sign for u to increase, E^+ is risk-averse if $b_1 > 0$ and risk-seeking if $b_1 < 0$,

whereas E^- is risk-averse if $b_1 < 0$ and risk-seeking if $b_1 > 0$. Although

neither E^+ nor E^- can be risk neutral, they can be arbitrarily close to risk

neutrality or linearity. For example, if $b_1 > 0$ in E^+ is set equal to

$c/(1 - e^{-b_2})$, then l'Hospital's rule shows that E^+ approaches cx and $b_1 b_2$

approaches c as b_2 goes to zero (and b_1 goes to $+\infty$). This shows that L is a

limiting case of E. It is therefore possible, for example, for the MMSE of

an L^+ fit to be smaller than the MSE's of all E^+ fits--and this must happen

when the positive data points lie on a straight line through the origin--but

it is then true also that there is no best E^+ fit and that the infimum of the

MSE values of the E^+ fits equals the MMSE of L^+.

Because the MMSE of a P fit can never exceed the MMSE of an L fit (and

the MMSE of an E fit, when it exists, can never exceed the MMSE of an L fit)

the best fit to the data among the three function types will be a power or

exponential fit. We shall therefore focus part of our attention later on the

question of whether power functions or exponential functions tend to yield the best fits over a number of situations. Although the later comparisons between P and E are purely a matter of fits to subjective empirical data, differences between the functions may help to place the results in perspective. For example, in the risk-averse P^+ and E^+ cases, the E^+ function is bounded above by b_1 while the P^+ function increases without bound, and Pratt's [21] local measure of absolute risk aversion $r(x) = -u''(x)/u'(x)$ is positive and constant for E^+ but positive and decreasing for P^+. For the risk-seeking cases in the positive region with $a_2 > 1$ or $b_1 < 0$, $r(x)$ is negative and constant for E^+ and negative and increasing for P^+, and although both functions grow rapidly the ratio of E^+ to P^+ approaches infinity as x gets large. The pictures for $x \leq 0$ are essentially reversed. Thus, risk-seeking for E^- and P^- arises with $a_2 < 1$ and $b_1 > 0$, with E^- in this case bounded below by $-b_1$.

Finally, it should be noted that P^+ and E^+ have a limiting form that is discontinuous at the origin with $u(0) = 0$ and $u(x) = k > 0$ for all $x > 0$. This arises from P^+ when $a_1 = k$ and a_2 goes to zero, and from E^+ when $b_1 = k$ and b_2 goes to infinity. This form, reflected into the negative quadrant, was used in one below-target case that is identified later as SA4 in Table 1A. This case consists of two below-target data points for which the one farther away from the target has the algebraically larger utility. Because of this there is no best P^- or E^- fit as such, but with k midway between the two utilities of the data points we can get the MSEs of P^- and E^- arbitrarily close to the MSE of the horizontal utility function whose value is k for all $x < 0$.

FITTING THE FUNCTIONS

As suggested earlier, the original data were transformed linearly so that the transformed target point and its utility were both zero. Because separate fits were made for the above-target data and the below-target data, we shall describe the transformations used for each, including the reverse transformation that describes the fit utility function in terms of the original data scales. In doing this we shall let (t, u_0) denote the target point and its utility in the original data format, and let (x,y) be the transformed data point that corresponds to the original data point (x_0,y_0). As before, $u(x)$ is the function fit to the transformed data, and $U(x_0)$ will be the utility function in the original data format that corresponds to $u(x)$.

The general linear transformations used for the $x_0 \geq t$ data were

$$x = k_1(x_0 - t), \quad y = k_2(y_0 - u_0) \text{ with } k_1, k_2 > 0.$$

Since $y_0 = u_0 + y/k_2$ and y is the transformed utility data value that is compared to the fit value $u(x)$, we get

$$U(x_0) = u_0 + u(k_1(x_0 - t))/k_2 \text{ for } x_0 \geq t$$

with derivative $U'(x_0) = (k_1/k_2)u'(x)$. The slopes of U at the target point for L^+ and E^+ are therefore $U'(t) = (k_1/k_2)c$ and $U'(t) = (k_1/k_2)b_1 b_2$ respectively.

The general linear transformations used for the $x_0 \leq t$ data were

$$x = k_3(t - x_0), \quad y = k_4(u_0 - y_0) \text{ with } k_3, k_4 > 0.$$

These transformations put the (x,y) pairs into the positive quadrant so that MMSE algorithms used in fitting functions to the above-target data could be used in precisely the same way to fit functions to the below-target data. In other words, the L^+, P^+ and E^+ forms at the outset of the preceding section were used for the (x,y) transformed data in the below-target cases. Given $u(x)$ as fit in this manner, it follows from $y_0 = u_0 - y/k_4$ that

$$U(x_0) = u_0 - u(k_3(t - x_0))/k_4 \quad \text{for } x_0 \leq t$$

with derivative $U'(x_0) = (k_3/k_4)u'(x)$. Note here that if $u_0 = 0$, $t = 0$ and $k_3 = k_4 = 1$, then $U(x_0) = -u(-x_0)$, which corresponds to our description of how the $x \leq 0$ functions were obtained from the $x \geq 0$ functions in the latter part of the first paragraph in the preceding section. In a below-target case the slopes of U at the target point for L^- and E^- are $U'(t) = (k_3/k_4)c$ and $U'(t) = (k_3/k_4)b_1 b_2$ respectively.

If we let c^+ be the MMSE value of c for the L^+ fit to the $x_0 \geq t$ data, let c^- be the MMSE value of c for the L^+ fit to the transformed $x_0 \leq t$ data, and let b_1^+, b_2^+, b_1^- and b_2^- have similar meanings for the exponential fits, then the ratio of $U'(t)$ for $x_0 \leq t$ to $U'(t)$ for $x_0 \geq t$ will be

$$R(t) = (k_2 k_3/k_1 k_4)(c^-/c^+) \quad \text{for the two-piece linear fit,}$$

$$R(t) = (k_2 k_3/k_1 k_4)(b_1^- b_2^-/b_1^+ b_2^+) \quad \text{for the two-piece exponential fit.}$$

We can of course use different functional forms below and above the target so that, for example, $R(t) = (k_2 k_3/k_1 k_4)(c^-/b_1^+ b_2^+)$ for the $L^- E^+$ case. In any event, values of $R(t) > 1$ suggest that utility rises more rapidly below the target than above the target, and values of $R(t) < 1$ suggest the converse.

Each below-target and above-target data set was transformed linearly as described above so that (10,10) was the most extreme data point. (In all cases except for the nonmonotonic below-target case SA4, it was fairly clear which point was the extreme point, with $(x,y) \leq (10,10)$ for all data points (x,y). The ensuing discussion does not pertain to the exceptional case.) It follows that every fit function began at (0,0) and went "towards" (10,10) but passed through the latter point only if the fit was exact at that point.

The restriction of $u(0) = 0$, or $U(t) = u_0$, limits the goodness of fit since it forces the function to pass through the indicated point. Better overall fits could obtain by not requiring $u(0)$ to equal zero--for example, by using $cx + d$ instead of cx for the linear fits--but the $u(0) = 0$ constraint for all functions above and below target ensures the continuity of each two-piece utility function at the target. Continuity at t could also be ensured without forcing $u(0)$ to equal zero, but best two-piece fits could only be obtained in this way by performing a simultaneous below-above fit. Since such a procedure would have been much more involved than the separated procedure it was not pursued.

The best c values for the linear cases were obtained by the usual differentiation method. The extreme transformed data-point value (10,10) was chosen to accommodate numerical computations for the nonlinear fits. We tried (1,1) and (100,100) also as extreme points and found these less well suited to the computer's processes. A standard Newton-Raphson technique was used in the nonlinear cases to estimate the values of the parameters that minimize the error sum of squares for the function being fit to the data. The Newton-Raphson minimizing procedure, which relies on the sum-of-squares function f being convex in the parameters, requires the selection of starting

values for a_1 and a_2 or b_1 and b_2. It then computes the gradient ∇f and Hessian inverse H^{-1}, iterating on the parameter vector p by $p^{i+1} = p^i - \alpha \nabla f(p^i) H^{-1}(p^i)$ with $0 < \alpha \leq 1$ until a near-zero gradient is reached.

Convergence rates for the Newton-Raphson procedure were sensitive to the starting values, which were chosen by visual examination of the data and selected trial computations, and to the value of the convergence factor α. Convergence was often facilitated by starting with α small (about .1) and increasing it towards 1.0 as the zero-gradient point was approached.

DATA ANALYSIS

The data involve 30 empirically-assessed utility functions from five sources. Each source is listed as follows along with the number of cases and the designation of these cases on Table 1.

Swalm [24]: 13 cases. Case SXk is for man k in group X.

Halter and Dean [12]: two cases, p. 64, for a grain farmer

(HDGF) and a college professor (HDCP).

Grayson [9]: 10 cases. Case GkAB is for individual A.B. on

page 30k; case Gxy is the case on page 3xy.

Green [10]: three cases. Case PGAB is for individual A.B.

Barnes and Reinmuth [2]: two cases, for contractors A (BRA) and

B(BRB).

Green's functions are based on percent return on investment; all others are based on dollar increments. We omitted one of Green's functions because of slightly ambiguous data, and omitted the Halter-Dean orchard farmer's function which appeared to be exactly linear below the target ($t = 0$) and above the target with $R(t) = 4$. All utility assessments included changes that involved many thousands of dollars.

The 30 cases are listed in Tables 1A and 1B. For each case we identify

Tables 1A and 1B about here

the target point, which was chosen by visual examination, and note how many data points other than the origin were used for the below-target fits and for the above-target fits. In all except Green's cases the data points were read from the plots as well as this could be done. Since Green presented only smooth functions without also showing his assessed data points, the data for his cases were taken directly from his curves at equally-spaced intervals of percent return.

All below-target and above-target cases had two or more data points other than the origin except for below-target cases G4FH and G8CS. No P and E fits were made for the latter two cases for obvious reasons. The above-target fits for these two situations had different curvatures: G4FH was convex and G8CS was concave.

Curvature

It follows that 28 of the 30 cases yield a composite below-target-- above-target picture for the curvatures of the P and E fits. Fortunately, it is unnecessary to separate P from E for present purposes since in each of the

TABLE 1A

Swalm (S) and Halter-Dean (HD) Cases

Case	Target	Shape	Below Target			Above Target			Slope Ratio R(t)	
			No. of Data Points	% of L^- P^-	MMSE: E^-	No. of Data Points	% of L^+ P^+	MMSE: E^+	L^-L^+	E^-E^+
SA1	0	1	6	49.5	74.3	8	4.55	6.67	14.8	9.3
SA6	0	1	4	91.2	93.1	7	80.5	91.7	2.1	2.2
SA27	0	1	6	88.1	97.2	7	9.65	14.1	1.8	1.1
SA34	0	4	3	32.8	18.9	8	71.7	62.6	3.6	6.9
SA10	0	3	5	87.1	83.5	6	4.95	6.90	1.7	.9
SA4	0	4	2	44.3	44.3	7	24.1	26.2	.8	∞
SB3	0	1	6	90.0	94.9	8	11.6	4.36	4.5	2.0
SB13	0	1	6	48.5	65.1	7	26.1	42.7	2.6	5.2
SB31	0	1	5	86.9	96.2	7	11.9	15.5	2.6	.8
SC18	0	1	5	13.0	1.30	8	13.8	10.9	3.4	1.7
SC22	0	1	6	66.2	79.9	8	8.80	13.7	7.7	11.0
SC24	0	1	4	11.2	.06	7	7.89	16.8	4.4	2.5
SC19	0	1	6	36.1	42.2	8	9.13	29.2	2.2	6.6
HDGF	0	1	4	6.12	1.94	4	11.3	1.88	5.0	5.5
HDCP	0	1	4	3.39	2.26	6	74.9	92.8	3.6	17.3

TABLE 1B

Grayson (G), Green (PG) and Barnes-Reinmuth (BR) Cases

Case	Target	Shape	Below Target No. of Data Points	% of MMSE: L^- P^-	E^-	Above Target No. of Data Points	% of MMSE: L^+ P^+	E^+	Slope Ratio R(t) L^-L^+	E^-E^+
G4JB	0	4	3	97.0	88.9	12	64.5	59.0	3.1	4.6
G4BB	-50	3	2	0	0	14	17.2	12.1	19.9	1.6
G4FH	-150		1	-	-	9	51.0	61.5	7.8	-
G6MR	0	3	3	3.92	2.01	9	99.3	96.6	19.8	.9
G8RH	0	2	3	18.1	31.9	5	4.17	15.2	5.5	6.5
G8JG	0	2	5	3.68	6.61	5	28.3	26.3	23.3	33000.
G8CS	-80		1	-	-	9	99.7	99.9	165.2	-
G10	0	2	4	.92	.12	8	80.5	78.2	4.8	2.2
G12	0	2	5	.46	.06	3	19.8	10.3	7.5	4.8
G13	-100	1	2	0	0	10	49.5	46.5	4.9	6.0
PGRM	20%	2	3	.02	1.88	4	35.7	16.3	5.2	1.7
PGWS	20%	2	3	8.13	3.45	4	30.4	12.7	2.9	.5
PGJB	20%	2	3	7.38	1.59	3	25.2	11.9	7.1	3.9
BRA	0	4	3	16.7	2.08	8	4.50	10.4	7.7	91.9
BRB	0	4	3	.08	8.05	6	78.5	71.1	24.7	81.7
		Means		32.5	33.6		35.3	35.5		
		Medians		14.8	7.3		24.7	16.5		
Better of P and E:		Means		29.6			29.0			
		Medians		3.6			12.4			

56 cases (28 below, 28 above) the P and E curvatures were identical in sign. The composite curvatures for the 28 cases are shown in the "Shape" column in Table 1, where 1 = (convex below, concave above), 2 = (concave below, convex above), 3 = (concave below and above) and 4 = (convex below and above). The summaries for these four types are as follows:

	concave above	convex above	
convex below	13	5	18
concave below	3	7	10
	16	12	28

This shows that the predominant composite is convex-concave (cf. Figure 1), followed by concave-convex. These two composite types cover more than 70 percent of the 28 cases, thus lending substantial support to the Kahneman-Tversky reflection effect [16], which in one form suggests that above-target risk aversion is often accompanied by below-target risk seeking, and that above-target risk seeking is often accompanied by below-target risk aversion.

It is also interesting to note that risk aversion both above and below the target was observed in only three of the 28 cases. This of course suggests the general untenability of the frequently-invoked assumption that individuals are everywhere risk averse.

In Table 2 we have separated the below-target cases from the above-target cases. Bypassing for the moment the question of whether P or E tends to give

=========================

Table 2 about here

=========================

TABLE 2

Curvature of Utility Functions and Best Fits

	Below Target				Above Target		
	P best	E best	P & E tied*		P best	E best	
Convex	9	7	2	18	4	9	13
Concave	3	6	1	10	11	6	17
	12	13	3	28	15	15	30

*Each of these three cases had two data points. One convex case (G13) and the concave case (G4BB) were monotonic, giving exact fits for P and E. The other convex case (SA4) was non-monotonic and gave the same limiting fit for P and E. The two below-target cases that had only one data point (G4FH and G8CS) are omitted from this table.

17

better fits, it will be noted that about 64 percent of the 28 below-target curves are convex and that about 57 percent of the 30 above-target curves are concave. Hence the majority trend below target is towards risk seeking, while the majority trend above target is towards risk aversion. It is also interesting to observe that below-target risk seeking is slightly more prevalent than above-target risk aversion although the figures are close enough to cast doubt on the generalizability of this observation.

Our final look at curvature _per se_ is provided by Figure 2, which plots the (a_1, a_2) pairs for the MMSE P fits for below-target cases (upper left) and above-target cases (upper right) along with the (b_1, b_2) pairs for the MMSE E fits for below-target cases (lower left) and above-target cases (lower right). In viewing this figure it should be recalled that the maximum data


Figure 2 about here
---

point for each above-target fit was transformed linearly into (10,10), and the minimum data point for each below-target fit was transformed linearly (with "reflection") into (10,10) also. The sets of parameter-pair points in Figure 2 do not lie along smooth curves (but come close) because the best fits did not necessarily pass through a common point (e.g. (10,10)) other than the origin.

Our primary interest in Figure 2 is the distributions of the curvature parameters, i.e. of a_2 for P and b_2 for E. Consider the power functions first. In the below-target P cases, seven of the a_2 values exceeded 2, seven were less than 1/2, and the remaining 14 fell between 1/2 and 2. In other words, only 50 percent of the below-target cases had a_2 values between the square-root

FIGURE 2
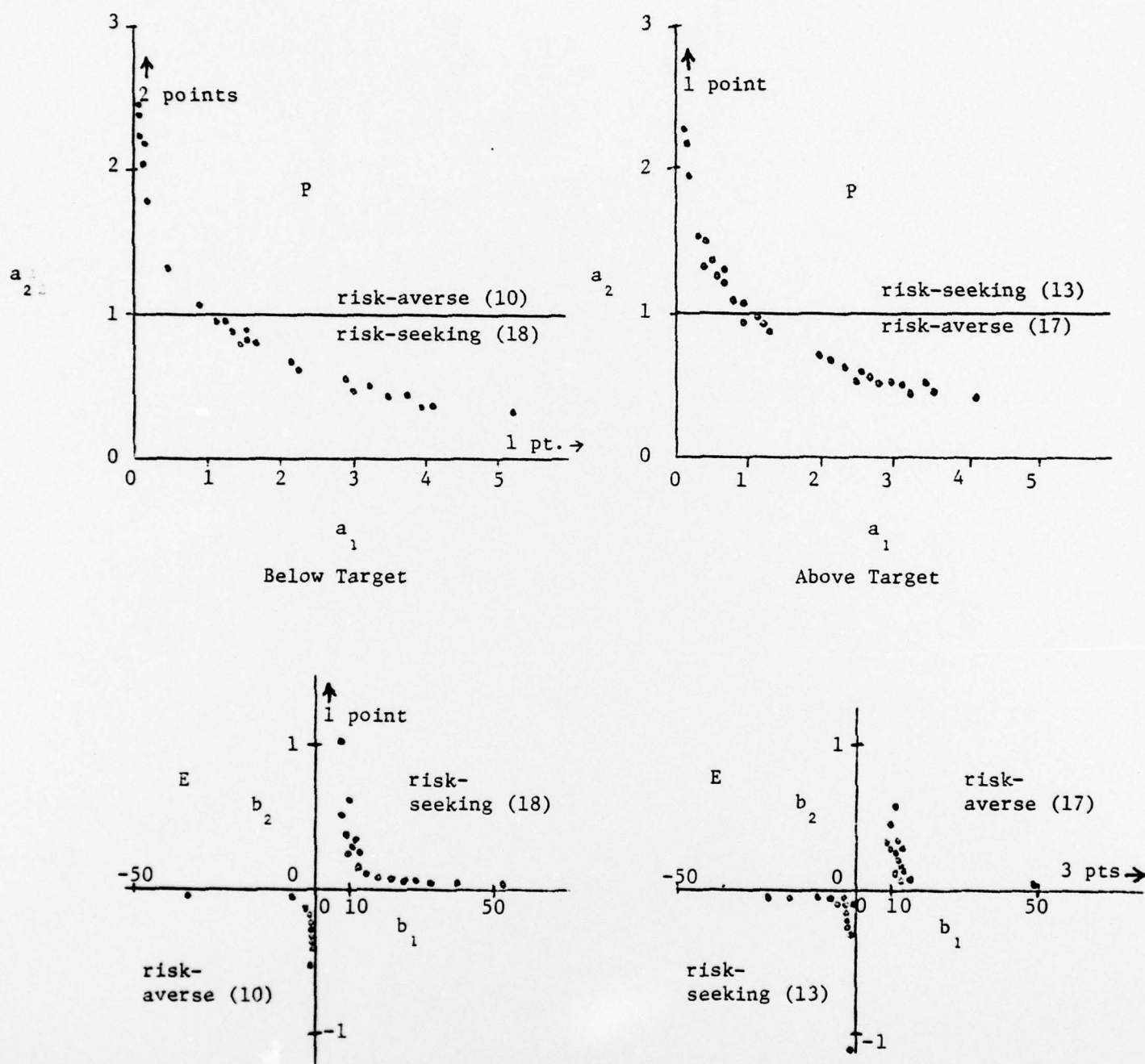
Parameter Pairs for Best Fits to Data

value of 1/2 and the square value 2. On the other hand, 80 percent of the a_2 values in the above-target cases fell between 1/2 and 2; the other 20 percent were split between > 2 and < 1/2. The obvious implication of these observations is that the below-target functions tend to exhibit more curvature than the above-target functions. Put another way, the above-target functions tend to be closer to linear functions than the below-target functions.

A similar pattern is observed for the exponential fits in the lower half of Figure 2, where larger $|b_2|$ values indicate more curvature. For example, 50 percent of the b_2 values for below-target E fits fell between -1/4 and +1/4, whereas 70 percent of the b_2 values for above-target E fits were between -1/4 and +1/4.

Changes in Slope

To examine the extent to which below-target utility increases more rapidly than above-target utility we computed the slope ratio R(t) of below-target slope to above-target slope for the two-piece linear fit L^-L^+ and the two-piece exponential fit E^-E^+. This was done for each of the 30 cases in Table 1 with the exception of E^-E^+ for cases G4FH and G8CS since below-target exponentials were not fit for these cases. The computations for R(t), whose results appear in the final two columns of Table 1, followed the procedure described earlier.

These computations strongly support the proposition that the utility function is steeper below the target than above the target. Cases to the contrary arose only once (SA4) in the 30 L^-L^+ cases and were observed four times (SA10, SB31, G6MR, PGWS) in the 28 applicable E^-E^+ cases. However, the larger value of R(t) exceeded unity in every instance. The median values of R(t) were about 4.9 for the two-piece linear fits and about 4.7 for the two-piece exponential fits.

Figure 3 displays the (R(t) for L^-L^+, R(t) for E^-E^+) pairs for the 28 applicable cases on Table 1. The absence of correlation between the two methods used to compute R(t) is shown by the fact that the Pearson product-

===========================
Figure 3 about here
===========================

moment correlation coefficient r is virtually zero for the 24 data points shown in the body of Figure 3. Although this finding was not fully anticipated, it does not seem unreasonable in view of the differences between L and E in the neighborhood of the target. If anything, the lack of correlation strengthens the steepness proposition since this proposition was found to hold for both methods. In other words, the assertion that below-target utility increases more rapidly than above-target utility is robust against two very different methods of computing this factor.

Goodness of Fits

We shall conclude our analysis with comparisons of P versus E and a discussion of the degree to which these functions give good fits to the data. The P versus E comparison will be considered first.

The bottom row of Table 2 shows that P and E were respectively best in 12 and 13 below-target cases and in 15 and 15 above-target cases. Therefore, neither P nor E significantly outperformed the other in either the below-target realm or the above-target realm. However, Table 2 does reveal one substantial difference between the two functions. To see this, we shall say that a function is <u>flat</u> if it is a convex below-target function or a concave above-target function (these functions flatten out as we move away from t) and that a function is <u>steep</u> if it is a concave below-target function or a

FIGURE 3
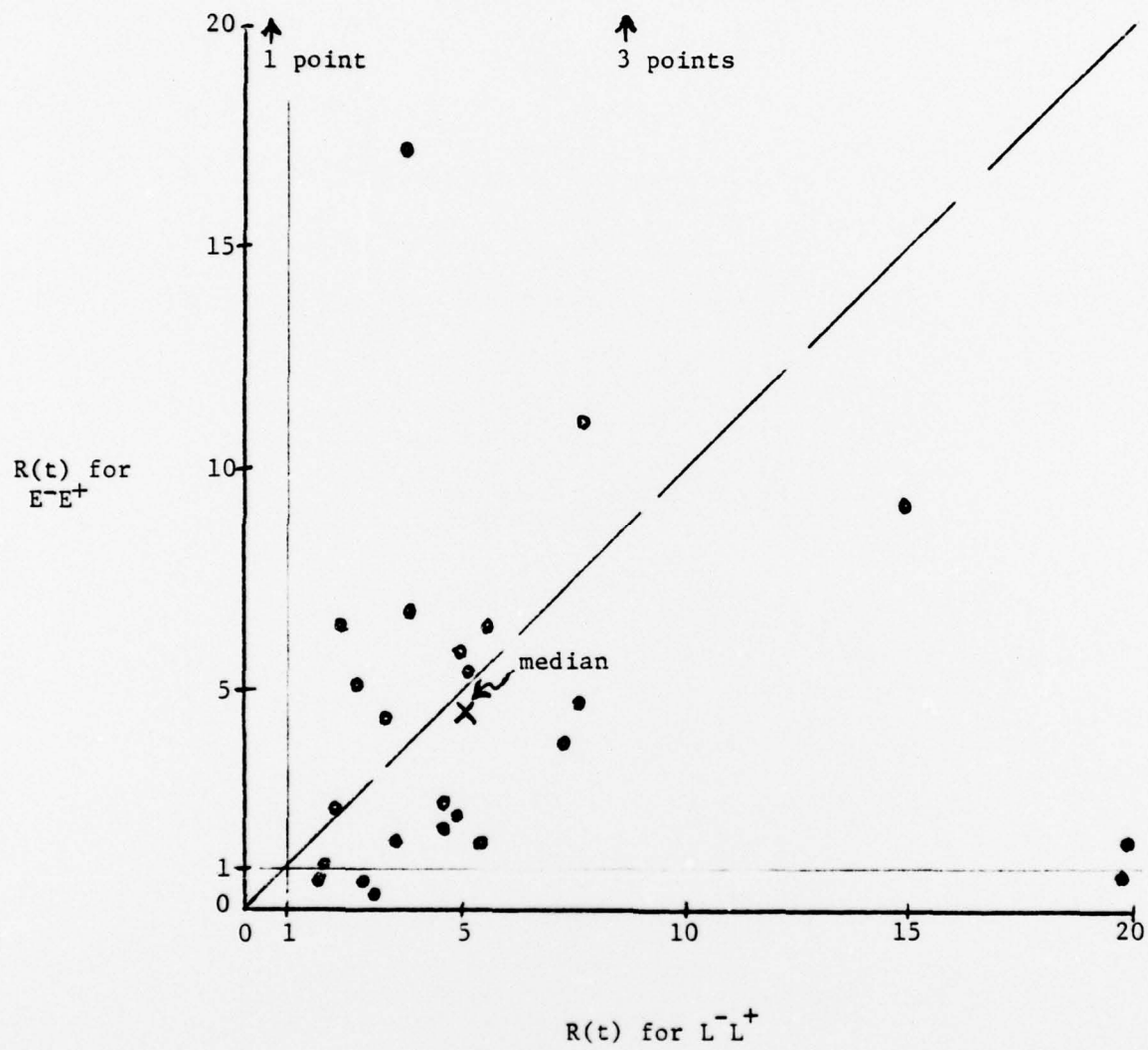
Pairs of Below-Target to Above-Target Slope Ratios

convex above-target function (these functions rise or drop rapidly as we move away from t). Table 2 then yields the following summary picture:

	P best	E best	
flat functions	20	13	33
steep functions	7	15	22
	27	28	55

Hence P gave a better fit than E in about 61 percent of the 33 flat cases, and E gave a better fit than P in about 68 percent of the 22 steep cases. Both figures are rather significant and permit the conclusion--in the context of the present data--that power functions tend to give better fits to flat data, whereas exponential functions tend to give better fits to steep data.

The MMSEs of the P and E fits are illustrated in two ways. Table 1 gives these values as percentages of the MMSEs for the linear fits in each case, and Figure 4 shows the (MMSE P, MMSE E) pair for each below-target case and each above-target case. Consider first the percentage data of Table 1. As shown at the bottom of the table, the means for the four columns (P and E, below and above) lie between 32 and 36 percent, but the median percentages are considerably smaller, varying from a low median of 7.3 percent for below-target E fits to a high of 24.7 percent for above-target P fits. There are seven below-target cases in which the smaller of the P and E MMSEs exceeds 50 percent of the linear MMSE, and nine above-target cases in which the smaller of the P and E MMSEs exceeds 50 percent of the linear MMSE. In most of these cases the fit P and E functions were close to linear (e.g. a_2 between .80 and 1.3 in 13 of these 16 poor-fit cases) which, along with an examination of the

data, suggests that significantly better fits would not be obtained with other simple functions.

The most encouraging percent-of-linear MMSE figures are shown in the last two lines of Table 1, where the larger means than medians reflect the high-percentage cases just noted and the preponderance of low-percentage cases. The median figures in the last line of the table show that in half of the below-target cases more than 96 percent of the linear MMSE was eliminated by the better of the P and E fits, and in half of the above-target cases more than 87 percent of the linear MMSE was eliminated by the better nonlinear fit.

Our final concern will be with the actual MMSE figures, which are shown on Figure 4 for the P and E fits. In viewing these figures it should be kept in mind that all fits were made to (x,y) pairs that ranged from (0,0) to (10,10). Hence an MMSE value of say .25 indicates that the average square of the distance between the fit function and the data point in the vertical direction for the case at hand is .25, which corresponds to an absolute vertical distance of .5 in a total range of 10.

Figure 4 about here

Figure 4 shows that the medians of the MMSE values for the P fits and for the E fits were both about .21. The corresponding median for the L fits was about 1.86. As might be expected from preceding discussion, the arithmetic means were somewhat larger than these medians. Because there are several ways of computing the means, we chose a conservative method in which the total sum of squares for all n cases is divided by the difference between

FIGURE 4
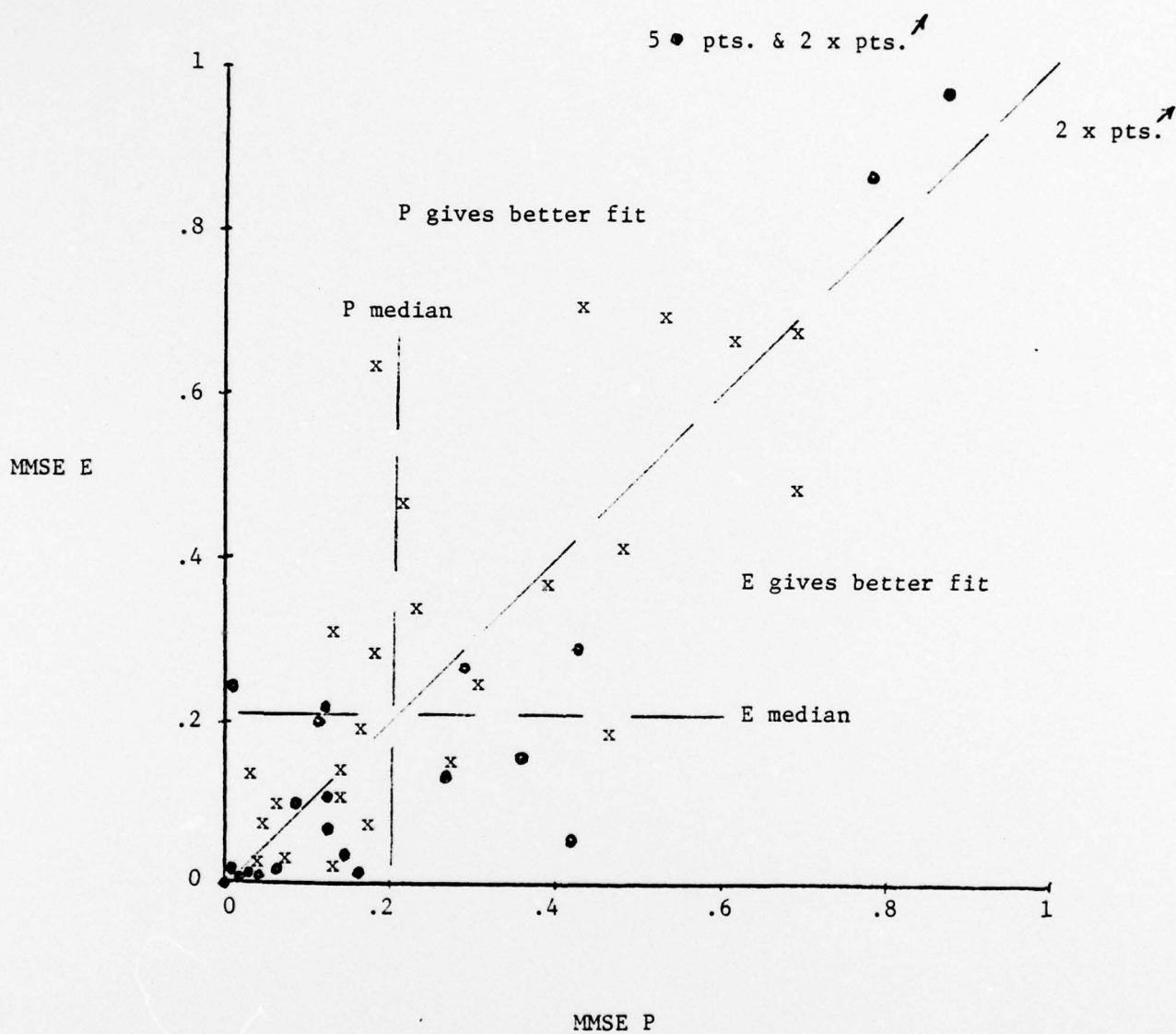
Plot of MMSE (P,E) Pairs for Below-Target (●)
and Above-Target (x) Cases

the total number of data points for the n cases and kn, where k is the number of parameters in the fitting function: k = 1 for L, k = 2 for P and E. The reason for subtracting kn from the number of data points is that a function with k parameters can always be fit exactly through k points, provided of course that conditions such as monotonicity for the P and E functions hold for the data. The nonmonotonic two-point below-target case SA4 was excluded from these computations, but all other applicable cases were included.

We shall refer to the mean values obtained by this procedure as the adjusted MSE averages. The below-target and above-target adjusted MSE averages along with the combined or pooled averages are shown in Table 3. The difference

Table 3 about here

between the averages for the below-target and above-target cases is apparently caused by the more erratic nature of the below-target data. Although we suspect that individuals find it more difficult to provide reliable utility assessments in the loss region than in the gain region, a host of other factors could affect this observed difference [9] [15] [17].

Because Table 3 is based on means instead of medians and on conservative means at that, its goodness-of-fit picture is not as rosy as that given earlier. Nevertheless, it does show that both the P and E fits are substantially better than the linear fits. When this way of viewing the data is combined with our preceding analysis, our own general impression is that the power and exponential functions provide fairly good fits to the assessment data.

TABLE 3

Adjusted MSE Averages			
	Below Target	Above Target	Pooled
Linear Fits	4.79	2.29	3.06
Power Fits	2.24	.66	1.08
Exponentials	2.78	.68	1.24
Best of P and E	2.17	.59	1.01

SUMMARY

Data from five sources for thirty empirically-assessed utility functions defined on changes in wealth or percent return on investment were analyzed for general trends and for their susceptability to representation by simple functional forms. Each of the thirty data sets was divided into below-target data and above-target data, and the functions were fit separately to each subset. In most cases the target was at the zero-gain point.

About two-thirds of the below-target functions were convex or risk-seeking, and slightly less than three-fifths of the above-target functions were concave or risk-averse. The predominant composite shape was convex below and concave above (46 percent). Of the other three composite types the concave-concave was observed least often (11 percent).

In essentially all cases, below-target utility was steeper than above-target utility. The median values of below-target slope divided by above-target slope as determined by two relatively uncorrelated methods were between 4 and 5.

Linear, power and exponential functions were fit to each data subset under the minimum-least-squares criterion. The power and exponential functions gave significantly better fits than the linear function even when the data were adjusted to account for the additional parameter in the nonlinear functions. Without this adjustment, the median minimum MSEs of the linear, power and exponential fits over all cases were respectively about 1.9, 0.2 and 0.2 for data sets whose minimum and maximum points were at $(0,0)$ and $(10,10)$. Power functions gave better fits than exponential functions in about three-fifths of the flat data sets (convex below or concave above), whereas exponential functions gave better fits than power functions in about two-thirds of the steep data sets (concave below or convex above).

REFERENCES

[1] Archibald, G. C. "Utility, Risk, and Linearity." Journal of Political Economy, Vol. 67, No. 5 (October, 1959), pp. 437-450.

[2] Barnes, J. D. and J. E. Reinmuth. "Comparing Imputed and Actual Utility Functions in a Competitive Bidding Setting." Decision Sciences, Vol. 7, No. 4 (October, 1976), pp. 801-812.

[3] Bernoulli, D. "Specimen Theoriae Novae de Mensura Sortis." Commentarii Academiae Scientiarum Imperialis Petropolitanae, Vol. 5 (1738), pp. 175-192.

[4] Bernoulli, D. "Exposition of a New Theory on the Measurement of Risk." [Translation of [3] by L. Sommer.] Econometrica, Vol. 22, No. 1 (January, 1954), pp. 23-36.

[5] Fishburn, P. C. Utility Theory for Decision Making. New York: John Wiley & Sons, 1970.

[6] Fishburn, P. C. "Mean-Risk Analysis with Risk Associated with Below-Target Returns." American Economic Review, Vol. 67, No. 2 (March, 1977), pp. 116-126.

[7] Friedman, M. and L. J. Savage. "The Utility Analysis of Choices Involving Risk." Journal of Political Economy, Vol. 56, No. 4 (August, 1948), pp. 279-304.

[8] Galanter, E. and P. Pliner. "Cross-Modality Matching of Money Against Other Continua." In H. R. Moskowitz, et al., eds. Sensation and Measurement. Dordrecht, Holland: Reidel Publishing Co., 1974, pp. 65-76.

[9] Grayson, C. J. Decisions Under Uncertainty: Drilling Decisions by Oil and Gas Operators. Cambridge, Mass.: Graduate School of Business, Harvard University, 1960.

[10] Green, P. E. "Risk Attitudes and Chemical Investment Decisions."
      Chemical Engineering Progress, Vol. 59, No. 1 (January, 1963), pp. 35-40.

[11] Hakansson, N. H. "Friedman-Savage Utility Functions Consistent with
      Risk Aversion." Quarterly Journal of Economics, Vol. 84, No. 3
      (August, 1970), pp. 472-487.

[12] Halter, A. N. and G. W. Dean. Decisions Under Uncertainty. Cincinnati,
      Ohio: South-Western Publishing Co., 1971.

[13] Hammond, J. S. "Better Decisions with Preference Theory." Harvard
      Business Review, Vol. 45, No. 6 (November-December, 1967), pp. 123-141.

[14] Herstein, I. N. and J. Milnor. "An Axiomatic Approach to Measurable
      Utility." Econometrica, Vol. 21 (April, 1953), pp. 291-297.

[15] Johnson, E. M. and G. P. Huber. "The Technology of Utility Assessment."
      IEEE Transactions on Systems, Man, and Cybernetics, Vol. SMC-7, No. 5
      (May, 1977), pp. 311-325.

[16] Kahneman, D. and A. Tversky. "Prospect Theory: An Analysis of Decision
      Under Risk." Econometrica (forthcoming).

[17] Keeney, R. L. and H. Raiffa. Decisions with Multiple Objectives:
      Preferences and Value Tradeoffs. New York: John Wiley and Sons, 1976.

[18] Kwang, N. Y. "Why Do People Buy Lottery Tickets? Choices Involving
      Risk and the Indivisibility of Expenditure." Journal of Political
      Economy, Vol. 73 (October, 1965), pp. 530-535.

[19] Libby, R. and P. C. Fishburn. "Behavioral Models of Risk Taking in
      Business Decisions: A Survey and Evaluation." Journal of Accounting
      Research, Vol. 15 (Autumn, 1977), pp. 272-292.

[20] Markowitz, H. "The Utility of Wealth." Journal of Political Economy,
      Vol. 60 (April, 1952), pp. 151-158.

[21] Pratt, J. W. "Risk Aversion in the Small and in the Large." *Econometrica*, Vol. 32, No. 1-2 (January-April, 1964), pp. 122-136.

[22] Rosett, R. N. "The Friedman-Savage Hypothesis and Convex Acceptance Sets: A Reconciliation." *Quarterly Journal of Economics*, Vol. 81 (1967), pp. 534-535.

[23] Slovic, P., B. Fischhoff, S. Lichtenstein, B. Corrigan and B. Combs. "Preference for Insuring Against Probable Small Losses: Insurance Implications." *Journal of Risk and Insurance*, Vol. 44, No. 2 (June, 1977), pp. 237-258.

[24] Swalm, R. D. "Utility Theory--Insights into Risk Taking." *Harvard Business Review*, Vol. 47 (November-December, 1966), pp. 123-136.

[25] von Neumann, J. and O. Morgenstern. *Theory of Games and Economic Behavior*, 2nd ed. Princeton, New Jersey: Princeton University Press, 1947.

[26] Yaari, M. E. "Convexity in the Theory of Choice Under Risk." *Quarterly Journal of Economics*, Vol. 79 (May, 1965), pp. 278-290.